UNIT - V

FINITE VOLUME METHODS

The most natural method to accomplish this is to discretize the integral form of the equations and not the differential form. This is the basis of a finite volume method.

The elements used in the finite element method is called as cells that cover the whole domain in the finite volume method.

The conservation laws are applied to determine the flow variables in some discrete points of the cells, called nodes.

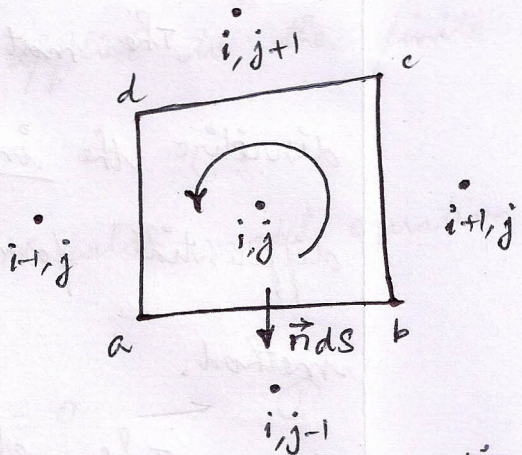
The grid of nodes on which pressure and density are defined is different from the grid of nodes on which velocity components x, y are defined. This approach commonly is called the staggered grid approach.

The finite volume method tries to combine the geometric flexibility from the finite element method and the flexibility in defining the discrete flow field values of dependent variables from the finite difference method.

Cell-centred formulation

The values of the dependent variables are stored in the centre of the cell as shown in figure.

These values are seen as mean values over the cell. Therefore, in the cell-centred method, values are attributed to the vertices of the grid by taking a weighted mean of the values in adjacent cells.



Cell-centred formulation

The set of Euler equations can be written in two dimensions as

$$\frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 \longrightarrow (1)$$

The semi-discretization of eqn (1) is obtained by.

$$\Omega_{i,j} \frac{\partial U}{\partial t} + \int_{abcd} \vec{F} \cdot \vec{n}\, dS = 0 \longrightarrow (2)$$

where,

$\Omega_{i,j}$ → control volume.

$\vec{F}$ → flux vector, $\vec{F} = f\vec{i}_x + g\vec{i}_y$.

dS → surface element

$\vec{n}$ → outward normal.

By taking the positive sense

$$\vec{n}\,ds = dy\,\vec{i}_x - dx\,\vec{i}_y \rightarrow (3)$$

Then eqn (2) becomes,

$$-\Omega_{i,j}\frac{\partial U}{\partial t} + \int_{abcd}(f\,dy - g\,dx) = 0 \rightarrow (4)$$

Further, f and g have to be defined on the boundary of the volume. A mean value between adjacent nodes are.

$$f_{ab} = \frac{f_{i,j} + f_{i,j-1}}{2} \;;\; g_{ab} = \frac{g_{i,j} + g_{i,j-1}}{2} \rightarrow (5)$$

Since the flux functions are non-linear functions of the dependent variables,

$$f_{ab} = f\left[\frac{U_{i,j} + U_{i,j-1}}{2}\right] \;;\; g_{ab} = g\left[\frac{U_{i,j} + U_{i,j-1}}{2}\right]$$

The above eqn is meant that the dependent variables are first averaged and that afterwards flux vectors are calculated. Since it implies about twice as many flux evaluations, eqn (5) is the best central flux definition.

Lax-Wendroff Time Stepping

In the finite difference method, Lax-Wendroff method is a very classic explicit time integration method. Since, the time-stepping can be applied to the finite volume formulation.

Let us consider the one-dimensional scalar model equation.

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \longrightarrow (1)$$

The Taylor series expansion to second order gives,

$$u^{n+1} \approx u^n + \left(\frac{\partial u}{\partial t}\right)^n \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)^n \frac{\Delta t^2}{2} \longrightarrow (2)$$

and

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right)$$

or

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial t}\right) \qquad \because a = \frac{\partial f}{\partial u}$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(a\frac{\partial f}{\partial x}\right) \longrightarrow (3)$$

Then eqn (2) is,

$$u^{n+1} \approx u^n - \left(\frac{\partial f^n}{\partial x}\right)\Delta t + \frac{\partial}{\partial x}\left(a\frac{\partial f^n}{\partial x}\right)\frac{\Delta t^2}{2} \longrightarrow (4)$$

The two dimensional analogue of eqn (4) is

$$U^{n+1} \approx U^n - \left(\frac{\partial f^n}{\partial x} + \frac{\partial g^n}{\partial y}\right)\Delta t + \left\{\frac{\partial}{\partial x}\left[A^n\left(\frac{\partial f^n}{\partial x} + \frac{\partial g^n}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[B^n\left(\frac{\partial f^n}{\partial x} + \frac{\partial g^n}{\partial y}\right)\right]\right\}\frac{\Delta t^2}{2} \quad \rightarrow (5)$$

Where A and B are the Jacobian matrices of the flux vectors:

$$A = \frac{\partial f}{\partial U} \quad ; \quad B = \frac{\partial g}{\partial U}$$

Since the definition of derivatives is not simple in the finite volume method, one-step methods are never used. The most popular two-step formulations, such as the Richtmyer variant and the MacCormack variant, can however be used without problems in the FVM.

In the MacCormack variant of the Lax-Wendroff method, eqn (2) is written as

$$u^{n+1} = \frac{u^n}{2} + \frac{\Delta t}{2}\left(\frac{\partial u}{\partial t}\right)^n + \frac{u^n}{2} + \frac{\Delta t}{2}\left[\frac{\partial}{\partial t}\left(u + \Delta t\frac{\partial u}{\partial t}\right)\right]^n \quad \rightarrow (6)$$

with predictor

$$\overline{u^{n+1}} = u^n + \left(\frac{\partial u}{\partial t}\right)^n \Delta t \quad \rightarrow (7)$$

eqn (6) can be written as corrector

$$u^{n+1} = \frac{1}{2}\left[u^n + \overline{u^{n+1}} + \frac{\partial}{\partial t}\overline{u^{n+1}}\,\Delta t\right] \rightarrow (8)$$

The discretization by Maccormack of eqns (7) & (8) is

$$\overline{u_i^{n+1}} = u_i^n - \left(\frac{f_{i+1}^n - f_i^n}{\Delta x}\right)\Delta t \rightarrow (9)$$

$$u_i^{n+1} = \frac{1}{2}\left[u_i^n + \overline{u_i^{n+1}} - \left(\frac{\overline{f_i^{n+1}} - \overline{f_{i-1}^{n+1}}}{\Delta x}\right)\Delta t\right] \rightarrow (10)$$

Equations (9) & (10) form the forward-backward variant.

Runge-Kutta Time Stepping (Multi-stage time stepping).

Runge Kutta time stepping schemes for ordinary differential equations are unstable when applied to the semi-discretization with the central flux.

$$\Omega_{i,j}\frac{\partial U}{\partial t} + \frac{1}{2}\left(\Delta y_{ab} f_{i,j-1} - \Delta x_{ab} g_{i,j-1}\right)$$

$$+\frac{1}{2}\left(\Delta y_{bc} f_{i+1,j} - \Delta x_{bc} g_{i+1,j}\right)$$

$$+\frac{1}{2}\left(\Delta y_{cd} f_{i,j+1} - \Delta x_{cd} g_{i,j+1}\right)$$

$$+\frac{1}{2}\left(\Delta y_{da} f_{i-1,j} - \Delta x_{da} g_{i-1,j}\right) = 0 \rightarrow (11).$$

There is no contribution of the central node in the flux balance in eqn ①, since the flux balance for a constant flux or a closed surface is zero.

The instability of Runge-Kutta time stepping can be seen by considering a Fourier analysis on a central space discretization of the eqn ① for the case of constant $a = \frac{\partial f}{\partial u}$

$$\frac{\partial u_i}{\partial t} = -a \frac{u_{i+1} - u_{i-1}}{2\Delta x} \longrightarrow ②$$

Inserting

$$u = z e^{j\omega x} = z e^{j\theta} \quad \therefore \theta = \omega x.$$

where,

$\omega \rightarrow$ wave number.

$j \rightarrow$ stands for $\sqrt{-1}$

Then.

$$z' = -a z \left(\frac{e^{j\theta} - e^{-j\theta}}{2\Delta x} \right) = -ja z \left(\frac{\sin\theta}{\Delta x} \right) \longrightarrow ③$$

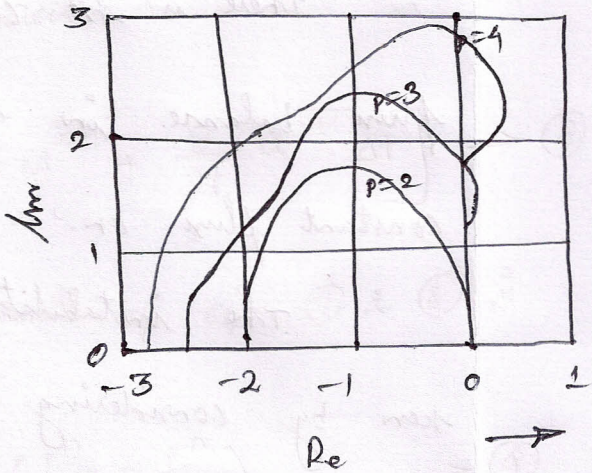
$$\therefore z' = \lambda z.$$

where,

$$\lambda = -ja \frac{\sin\theta}{\Delta x} \longrightarrow ④.$$

Figure shows the stability domain for the Runge-Kutta second, third and fourth order, time-integration methods.

Since λ is on the imaginary axis, the second order Runge-Kutta method is unstable. Higher order Runge-Kutta methods are marginally stable. Higher order Runge-Kutta methods can be stabilized by introducing a small amount of artificial viscosity.



Stability regions in the complex plane for classic explicit Runge-Kutta methods

Then Eqn (2) is,

$$\frac{\partial u}{\partial t} = -a \frac{u_{i+1} - u_{i-1}}{2\Delta x} + \varepsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \rightarrow (5)$$

Then,

$$\lambda = -ja \frac{\sin\theta}{\Delta x} - \frac{2\varepsilon}{\Delta x^2}(1 - \cos\theta) \rightarrow (6)$$

Since, there is now a small negative real part in λ, higher order Runge-Kutta time-stepping now becomes stable, when subject to a CFL-condition which restricts the time step.

The modification of second order derivative to fourth order derivative will lead to a similar stabilization effect.

Accuracy.

The representation of the solution is done in a piecewise constant way, on an irregular grid the accuracy is formally of first order. In practice, the order is between one to two.

Cell-Vertex Formulation.

In the cell-vertex formulation, the variables are stored at the vertices of the grid. The control volumes either coincide with cells or consist of a group of cells around a node. The cell-vertex formulations have the possibility to be second-order accurate in space, irrespective of the irregularity of the grid.

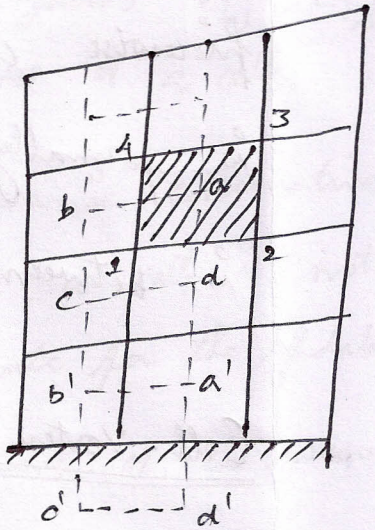
Multi-stage Time stepping (overlapping control volumes)

For the overlapping control volumes, the semi-discretization is very similar to the following eqn.

$$\Omega_{i,j}\frac{\partial U}{\partial t} + \frac{1}{2}\left(\Delta y_{ab} f_{i,j-1} - \Delta x_{ab} g_{i,j-1}\right)$$
$$+\frac{1}{2}\left(\Delta y_{bc} f_{i+1,j} - \Delta x_{bc} g_{i+1,j}\right) + \frac{1}{2}\left(\Delta y_{cd} f_{i,j+1} - \Delta x_{cd} g_{i,j+1}\right)$$
$$+\frac{1}{2}\left(\Delta y_{da} f_{i-1,j} - \Delta x_{da} g_{i-1,j}\right) = 0.$$

now involving, six or eight surrounding nodes. At solid boundaries, half volumes are formed.

The impermeability can be expressed by setting the convective fluxes to zero. Another approach is to treat the control volume as permeable and to impose tangency. This means that, between steps, the normal component of velocity is set equal to zero.



Quadrilateral cells, non-overlapping volumes.

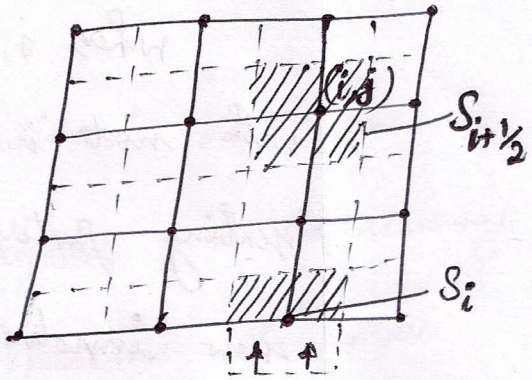
In order to stabilize the scheme, some form of artificial viscosity is necessary and is also necessary to eliminate the spurious modes in the solution.

Often, the dissipative operator of the cell-centred method is used. This operator is then a sum of terms of following equation for a quadrilateral grid.

$$d_{i+\frac{1}{2}j} = \varepsilon^{(2)}_{i+\frac{1}{2}j}\left(U_{i+1,j} - U_{i,j}\right) - \varepsilon^{(4)}_{i+\frac{j}{2}}\left(U_{i+2,j} - 3U_{i+1,j} + 3U_{i,j} - U_{i-1,j}\right)$$

The method loses then its pure cell-vertex character. The resulting flux balance of inviscid and dissipative terms is then a balance over a control volume centred around a vertex as shown in following figure.

Such a control volume is called a dual control volume. The inviscid flux balance over the dual control volume can be defined as one fourth of the flux balance over the volume formed by the four surrounding cells. Strictly, the method then becomes a vertex-centred or vertex-based method.



Vertex based FVM.

A pure cell-vertex method can be obtained by changing the construction of the dissipator.

W.R.T, the dissipation term.

$$D_{i,j} = \sigma_{i,j}\left[\varepsilon^{(2)}_{i,j}(U_j - U_i) - \varepsilon^{(4)}_{i,j}(\Delta U_j - \Delta U_i)\right].$$

The same methodology as for the cell-centred method is used, but summations now run over cells surrounding a node rather than over surrounding nodes.

For instance, $U_j - U_i$ is to be replaced by.

$$\frac{1}{2}(U_{j_1} + U_{j_2}) - U_i \quad \text{(triangular)}$$

or

$$\frac{1}{3}(U_{j_1} + U_{j_2} + U_{j_3}) - U_i \quad \text{or} \quad \frac{1}{2}(U_{j_1} + U_{j_3}) - U_i$$

(quadrilateral).

where, j_1, j_2 and j_3 denote the nodes not coinciding with node i of the ~~two~~ surrounding cells. Also the scaling factors $\sigma_{i,j}$ and the weight factors $\varepsilon^{(2)}_{i,j}$, $\varepsilon^{(4)}_{i,j}$ now involve maxima over all nodes of a cell.

The foregoing smoothing procedure is conservative in the sense that the content of a cell is not changed by the dissipator. The formula for the update of a node is the sum of contributions of the surrounding cells.

The modification is such that the flux balance over a cell can be seen as distributed to its vertices in an unequal way but with a sum of distribution factors equal the one. So, the dissipator acts as a redistributor of the flux balances of the cells.

Lax-Wendroff Time-stepping (Non-overlapping Control Volumes).

For the non-overlapping case, a Lax-Wendroff variant exists. It requires the use of a second set of control volumes centred around the nodes.

The Taylor series expansion the second order gives

$$u^{n+1} \approx u^n + \left(\frac{\partial u}{\partial t}\right)^n \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)^n \frac{(\Delta t)^2}{2} \rightarrow (1)$$

from the one-dimensional scalar model equation.

$$\frac{\partial u}{\partial t} = -\frac{\partial f}{\partial x} \longrightarrow (2)$$

Then,

$$u^{n+1} \simeq u^n - \left(\frac{\partial f}{\partial x}\right)^n \Delta t - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right)^n \frac{(\Delta t)^2}{2} \longrightarrow (3)$$

where, $\frac{\partial f}{\partial t}$ can be replaced by a first-order accurate difference $\frac{\Delta f}{\Delta t}$. The result is

$$u^{n+1} \simeq u^n - \left(\frac{\partial f}{\partial x}\right)^n \Delta t - \frac{\partial}{\partial x}(\Delta f)\frac{\Delta t}{2} \longrightarrow (4)$$

In two dimensions, on the Euler equations, this is

$$\Omega_{i,j}(U^{n+1} - U^n) = -\Delta t\left[\int (f^n dy - g^n dx) + \frac{1}{2}\int (\Delta f\, dy - \Delta g\, dx)\right] \longrightarrow (5)$$

On the quadrilateral grid, based on the cell 1-2-3-4, using a step forward in time, a first-order approximation of the increment of the flux vectors is obtained from

$$\Omega_a \Delta U_a = -\Delta t \int_{1234} (f\,dy - g\,dx)^n \longrightarrow (6)$$

and

$$\Delta f_a = A\,\Delta U_a \;;\; \Delta g_a = B\,\Delta U_a.$$

where A and B are the Jacobians of the flux vectors f and g with respect to U.

The area-weighted mean value of the first order increments given by above eqn (6) $\Omega_a \Delta U_a$ over the four cells surrounding the node 1, gives a first-order increment for the dependent variables ΔU_1^1.

The discretization of two dimensional Euler equations on the cell abcd is then:

$$\Omega_1 \left(U_1^{n+1} - U_1^n \right) = \Omega_1 \Delta U_1^1 - \frac{\Delta t}{2} \int_{abcd} (\Delta f \, dy - \Delta g \, dx) \rightarrow (7)$$

The spatial integration is again taken to be piecewise linear.

The CFL-restriction for the time step,

$$\Delta t \leq \min \left(\frac{\Delta x}{|u|+c} , \frac{\Delta y}{|u|+c} \right)$$

with,

$$\Delta x = \frac{x_{i+1,j} - x_{i-1,j}}{2} , \quad \Delta y = \frac{y_{i,j+1} - y_{i,j-1}}{2} .$$

The boundary conditions at solid solid boundaries for the first step eqn (6) can be implemented by setting convective fluxes equal to zero. In the second step eqn (7), a half-volume is needed around a boundary node.

FDM-like finite volume methods.

A cell-centred FVM is less attractive than FDM in which the nodes are given at the vertices of the grid. In a cell-vertex FVM, the flux through a volume surface is continuous.

More freedom in the definition of a flux, combined with nodes at the vertices of the grid, can be obtained by using an interweaving grid. The interweaving grid can be constructed by connecting the cell-centres.

Fluxes at volume faces can be defined as average of fluxes calculated with function values in adjacent nodes. The semi-discretization is then very close to a finite difference semi-discretization and can be called a conservative finite difference method.

Central type discretization

The adaptation of the Lax-Wendroff time-stepping or the multi-stage time-stepping to the vertex-based FVM is akin to the cell-centred FVM. The formulations obtained with both methods are very similar, except at solid boundaries.

Upwind type discretization

The upwind discretization can be explained by treating the flux-difference splitting technique. The flux through a surface $(i+\frac{1}{2})$ of the control volume can be written as

$$F_{i+\frac{1}{2}} = \Delta y_{i+\frac{1}{2}} f_{i+\frac{1}{2}} - \Delta x_{i+\frac{1}{2}} g_{i+\frac{1}{2}} \longrightarrow (1)$$

where,

$f_{i+\frac{1}{2}}$, $g_{i+\frac{1}{2}}$ — flux vectors in the nodes (i,j) and $(i+1,j)$.

consider halves in the subscripts to denote intermediate points and non-varying subscripts are not written.

The eqn (1) can be written as,

$$F_{i+\frac{1}{2}} = \Delta S_{i+\frac{1}{2}} \left(n_x f_{i+\frac{1}{2}} + n_y g_{i+\frac{1}{2}} \right) \longrightarrow (2)$$

with,

$$n_x = \frac{\Delta y_{i+\frac{1}{2}}}{\Delta S_{i+\frac{1}{2}}} \quad ; \quad n_y = \frac{-\Delta x_{i+\frac{1}{2}}}{\Delta S_{i+\frac{1}{2}}}.$$

$$\Delta S^2_{i+\frac{1}{2}} = \Delta x^2_{i+\frac{1}{2}} + \Delta y^2_{i+\frac{1}{2}}.$$

In order to define an upwind flux, consider the flux difference between i & i+1.

$$\Delta F_{i,i+1} = \Delta S_{i+1/2} \left(n_x \Delta f_{i,i+1} + n_y \Delta g_{i,i+1} \right) \rightarrow (3')$$

where,

$$\Delta f_{i,i+1} = f_{i+1,j} - f_{i,j} \quad , \quad \Delta g_{i,i+1} = g_{i+1,j} - g_{i,j}$$

For construction of the flux, it is essential that the linear combination of Δf and Δg in (3') can be written as

$$\Delta \phi = n_x \Delta f + n_y \Delta g = A \Delta U \rightarrow (4')$$

where,

$A \rightarrow$ discrete Jacobian matrix of the flux vectors.

The eigenvalues of A are real and that the matrix has a complete set of eigen vectors. The construction of the discrete Jacobian is not unique.

The matrix A can be split into positive and negative parts by

$$A^+ = R \Lambda^+ L \quad ; \quad A^- = R \Lambda^- L \rightarrow (5)$$

where,

$R \rightarrow$ Right eigenvector matrix in orthonormal form.

$L \rightarrow$ Left eigenvector matrix in orthonormal form.

and,

$$\Lambda^+ = diag(\lambda_1^+, \lambda_2^+, \lambda_3^+, \lambda_4^+) \; ; \quad \Lambda^- = diag(\lambda_1^-, \lambda_2^-, \lambda_3^-, \lambda_4^-)$$

with,

$$\lambda_i^+ = max(\lambda_i, 0) \quad ; \quad \lambda_i^- = min(\lambda_i, 0).$$

Then eqn (4) is written as,

$$\Delta \phi = A^+ \Delta U + A^- \Delta U$$

As a consequence eqn (3) can be written as

$$\Delta F_{i,i+1} = F_{i+1} - F_i = \Delta S_{i+1/2} \; A_{i,i+1} \; \Delta U_{i,i+1}$$

where,

matrix $A_{i,i+1}$ → Difference of positive and negative parts.

The absolute value of the flux-difference is defined by

$$|\Delta F_{i,i+1}| = \Delta S_{i+1/2} \left(A^+_{i,i+1} - A^-_{i,i+1}\right) \Delta U_{i,i+1} \rightarrow (6)$$

Based on eqn (6) an upwind definition of the flux is

$$F_{i+1/2} = \frac{1}{2}\left[F_i + F_{i+1} - |\Delta F_{i,i+1}|\right] \longrightarrow (7)$$

The eqn (7) can be verified by following two ways which are completely equivalent.

$$F_{i+1/2} = F_i + \frac{\Delta F_{i,i+1}}{2} - \frac{1}{2}\left|\Delta F_{i,i+1}\right| = F_i + \Delta S_{i+1/2} A^-_{i,i+1} \Delta U_{i,i+1}$$

→ (8)

$$F_{i+1/2} = F_{i+1} - \frac{\Delta F_{i,i+1}}{2} - \frac{1}{2}\left|\Delta F_{i,i+1}\right| = F_{i+1} - \Delta S_{i+1/2} A^+_{i,i+1} \Delta U_{i,i+1}$$

→ (9)

The fluxes on the other surfaces of the control volume $S_{i-1/2}$, $S_{j+1/2}$, $S_{j-1/2}$ can be treated in a similar way as the flux on the surface $S_{i+1/2}$. With eqn (8) and eqn (9), the flux balance on the control volume can be brought into the form.

$$\Delta S_{i+1/2} A^-_{i,i+1}\left[U_{i+1} - U_i\right] + \Delta S_{i-1/2} A^+_{i,i-1}\left[U_i - U_{i-1}\right]$$

$$+ \Delta S_{j+1/2} A^-_{j,j+1}\left[U_{j+1} - U_j\right] + \Delta S_{j-1/2} A^+_{j,j-1}\left[U_j - U_{j-1}\right] = 0$$

→ (10)

By keeping $U_{i,j}$ terms in left hand side,

$$C U_{i,j} = \Delta S_{i+1/2}\left(-A^-_{i,i+1}\right) U_{i+1,j} + \Delta S_{i-1/2} A^+_{i,i-1} U_{i-1,j}$$

$$+ \Delta S_{j+1/2}\left(-A^-_{j,j+1}\right) U_{i,j+1} + \Delta S_{j-1/2} A^+_{j,j-1} U_{i,j-1}$$

→ (11)

where,

C → Sum of the matrix-coefficients on the RHS.

The matrix coefficients of eqn (11) have non-negative eigenvalues.

Flux-Vector Splitting.

The definition of the eigenvalues λ_j of the matrix A is given by

$$|A - \lambda I| = 0 \longrightarrow (1)$$

The eigenvector associated with a specific eigenvalue λ_j can be defined as the column vector L_j which is a solution of the equation

$$[L_j]^T [A - \lambda_j I] = 0. \longrightarrow (2)$$

where,

$[L_j]^T \rightarrow$ the transverse of the column vector. Hence it is a row vector.

Since A and λ_j are known in eqn (2), the elements of L_j are obtained directly by solving eqn (2). For each different eigenvalue of the matrix A, there will be a different eigenvector L_j.

Let us now define a matrix T whose inverse T^{-1} has elements that are the elements of all the eigenvectors. Specifically, the j^{th} row of T^{-1} consists of the elements of the left eigenvector for λ_j.

Since $[L_j]^T$ appears on the left of eqn (2), L_j is called a <u>left eigenvector</u> of the matrix A. Now, the matrix T has the property of "diagonalizing" the matrix A through the equation

$$T^{-1} A T = [\lambda] \longrightarrow (3)$$

where,

$[\lambda]$ → diagonal matrix with the eigenvalues of A as the diagonal terms.

For instance,

$$[\lambda] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \longrightarrow (4)$$

Multiplying the matrix eqn (3), first by T on the left of both sides and then by T^{-1} on the right of both sides.

$$A = T[\lambda] T^{-1} \longrightarrow (5)$$

Consider the Euler equations for unsteady, one-dimensional flow

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \longrightarrow (6)$$

W.K.T A is the Jacobian of F,

$$A = \frac{\partial F}{\partial U}.$$

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial U}\frac{\partial U}{\partial x} = 0.$$

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0.$$

For an inviscid flow, the flux vector F can be expressed directly in terms of its Jacobian as,

$$F = AU \longrightarrow (7)$$

Let us define two matrices $[\lambda^+]$ and $[\lambda^-]$ made up of the positive and negative eigenvalues of A.

For subsonic flow,

$$\left.\begin{aligned} \lambda_1 &= u \\ \lambda_2 &= u + c \end{aligned}\right\} \text{- Positive Value.}$$

$$\lambda_3 = u - c \quad \text{- negative value.}$$

Therefore, by definition

$$[\lambda^+] = \begin{bmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$[\lambda^-] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & u-c \end{bmatrix}$$

from eqn (5), we can define A^+ and A^- as.

$$A^+ = T[\lambda^+]T^{-1} \longrightarrow (8)$$

and,

$$A^- = T[\lambda^-]T^{-1} \longrightarrow (9)$$

with this, we can split the flux vector F into two parts,

$$F = F^+ + F^- \longrightarrow (10)$$

then, from eqn (9),

$$F^+ = A^+ U \longrightarrow (11)$$

$$F^- = A^- U \longrightarrow (12)$$

Hence, eqn (6) can be written as,

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F^+}{\partial x} + \frac{\partial F^-}{\partial x} = 0} \longrightarrow (13).$$

The eqn (13) is an example of flux-vector splitting.

The Pressure correction formula.

The pressure correction p' can be constructed using the continuity equation.

$$p = p^* + p' \longrightarrow (1)$$

where,

$p \rightarrow$ corrected pressure.

$p^* \rightarrow$ Guessed pressure.

$p' \rightarrow$ Pressure correction.

consider a two dimensional flow and neglect body forces. The x and y momentum equations for an incompressible viscous flow in conservation form are,

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

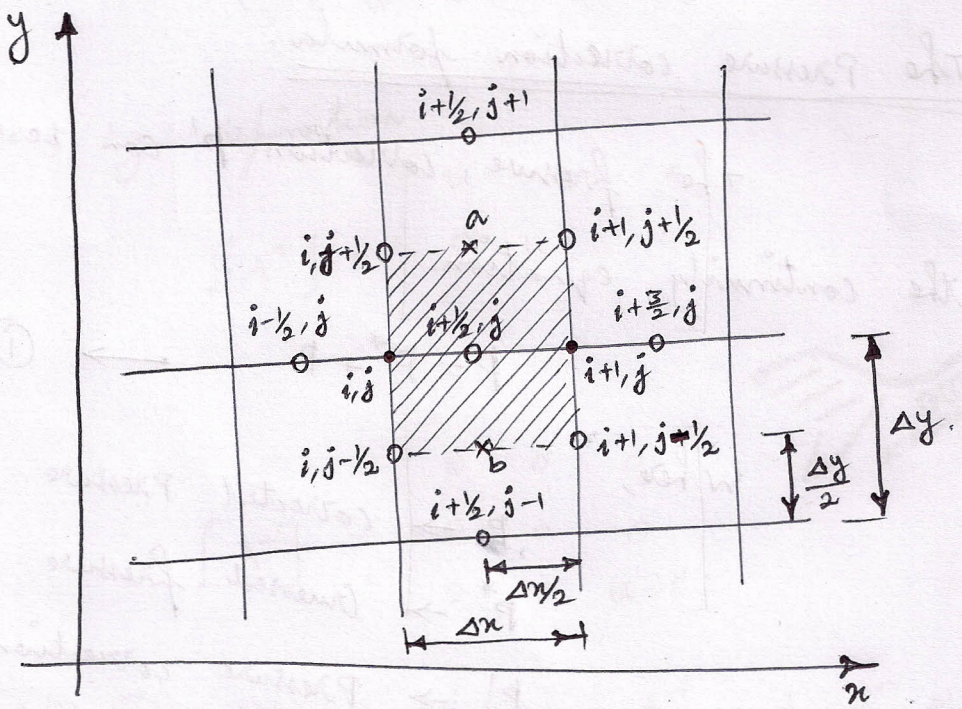
$\longrightarrow (2)$

and

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \rightarrow (3)$$

We proceed to develop the discretized equations which are the basic tools of the pressure correction method in which forward difference in time and central differences for the spatial derivatives are taken.

Consider a region in a staggered grid.



computational module for the x-momentum equation.

The pressures are evaluated at the solid grid points and the velocities at the open grid points. velocities at a & b are,

at point a ; $v_{j+1/2} = \frac{1}{2}\left(v_{i,j+1/2} + v_{i+1,j+1/2}\right)$

at point b ; $v_{j-1/2} = \frac{1}{2}\left(v_{i,j-1/2} + v_{i+1,j-1/2}\right)$

centred around point $(i+\frac{1}{2}, j)$ a difference representation of eqn (2) is

$$\frac{(\rho u)^{n+1}_{i+\frac{1}{2},j} - (\rho u)^{n}_{i+\frac{1}{2},j}}{\Delta t} = -\left[\frac{(\rho u^2)^{n}_{i+\frac{3}{2},j} - (\rho u^2)^{n}_{i-\frac{1}{2},j}}{2\Delta x}\right] - \frac{p^{n}_{i+1,j} - p^{n}_{i,j}}{\Delta x}$$

$$- \left[\frac{(\rho uv)^{n}_{i+\frac{1}{2},j+1} - (\rho uv)^{n}_{i+\frac{1}{2},j-1}}{2\Delta y}\right]$$

$$+\mu\left[\frac{u^{n}_{i+\frac{3}{2},j} - 2u^{n}_{i+\frac{1}{2},j} + u^{n}_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{u^{n}_{i+\frac{1}{2},j+1} - 2u^{n}_{i+\frac{1}{2},j} + u^{n}_{i+\frac{1}{2},j-1}}{\Delta y^2}\right] \rightarrow (4)$$

where,

$$-\left[\frac{(\rho u^2)^{n}_{i+\frac{3}{2},j} - (\rho u^2)^{n}_{i-\frac{1}{2},j}}{2\Delta x} + \frac{(\rho uv)^{n}_{i+\frac{1}{2},j+1} - (\rho uv)^{n}_{i+\frac{1}{2},j-1}}{2\Delta y}\right]$$

$$+\mu\left[\frac{u^{n}_{i+\frac{3}{2},j} - 2u^{n}_{i+\frac{1}{2},j} + u^{n}_{i-\frac{1}{2},j}}{\Delta x^2} + \frac{u^{n}_{i+\frac{1}{2},j+1} - 2u^{n}_{i+\frac{1}{2},j} + u^{n}_{i+\frac{1}{2},j-1}}{\Delta y^2}\right] = A.$$

Then eqn (4) can be written as

$$(\rho u)^{n+1}_{i+\frac{1}{2},j} = (\rho u)^{n}_{i+\frac{1}{2},j} + A\Delta t - \frac{\Delta t}{\Delta x}\left(p^{n}_{i+1,j} - p^{n}_{i,j}\right) \rightarrow (5)$$

III^{ly} in y- direction.

$$(\rho v)^{n+1}_{i,j+1/2} = (\rho v)^{n}_{i,j+1/2} + B\Delta t - \frac{\Delta t}{\Delta y}\left(P^{n}_{i,j+1} - P^{n}_{i,j}\right) \quad \rightarrow (6)$$

Eqns (5) & (6) are difference equations representing the x and y momentum equations respectively.

At the beginning of new iteration, $P = P^*$. Then eqn (5) & (6) becomes

$$(\rho u^*)^{n+1}_{i+1/2,j} = (\rho u^*)^{n}_{i+1/2,j} + A^*\Delta t - \frac{\Delta t}{\Delta x}\left(P^*_{i+1,j} - P^*_{i,j}\right)^n \quad \rightarrow (7)$$

and,

$$(\rho v^*)^{n+1}_{i,j+1/2} = (\rho v^*)^{n}_{i,j+1/2} + B^*\Delta t - \frac{\Delta t}{\Delta y}\left(P^*_{i,j+1} - P^*_{i,j}\right)^n \quad \rightarrow (8)$$

W.k.t from eqn (1)

$$\boxed{P - P^* = P'}$$

Then subtraiting eqn (7) from (5) gives.

$$(\rho u')^{n+1}_{i+1/2,j} = (\rho u')^{n}_{i+1/2,j} + A'\Delta t - \frac{\Delta t}{\Delta x}\left(P'_{i+1,j} - P'_{i,j}\right)^n \quad \rightarrow (9)$$

III^{ly} for eqn (8) & (6).

$$(\rho v')^{n+1}_{i,j+1/2} = (\rho v')^{n}_{i,j+1/2} + B'\Delta t - \frac{\Delta t}{\Delta y}\left(P'_{i,j+1} - P'_{i,j}\right)^n \quad \rightarrow (10)$$

Eqns (9) & (10) are the x and y momentum equations in terms of the pressure and velocity corrections.

The velocity field must satisfy the continuity equation to obtain a formula for the pressure correction p'.

For which we are concerned with two aspects.

i) The formula for p' must yield the values that ultimately lead to the proper, converged solution.

ii) In the limit of the converged solution, the formula for p' must reduce to the physically correct continuity equation.

Let us arbitrarily set A', B', $(\rho u')^n$ and $(\rho v')^n$ equal to zero in eqns (9) & (10).

$$(\rho u')^{n+1}_{i+\frac{1}{2},j} = -\frac{\Delta t}{\Delta x}\left(p'_{i+1,j} - p'_{i,j}\right)^n \longrightarrow (11)$$

and

$$(\rho v')^{n+1}_{i,j+\frac{1}{2}} = -\frac{\Delta t}{\Delta y}\left(p'_{i,j+1} - p'_{i,j}\right)^n \longrightarrow (12)$$

$$\boxed{p' = p - p^*}$$

Returning to eqn (7) brings the (11) as

$$(\rho u)^{n+1}_{i+\frac{1}{2},j} = (\rho u^*)^{n+1}_{i+\frac{1}{2},j} - \frac{\Delta t}{\Delta x}\left(p'_{i+1,j} - p'_{i,j}\right)^n \longrightarrow (13)$$

and eqn (12) becomes,

$$(\rho v)^{n+1}_{i,j+\frac{1}{2}} = (\rho v^*)^{n+1}_{i,j+\frac{1}{2}} - \frac{\Delta t}{\Delta y}\left(P'_{i,j+1} - P'_{i,j}\right)^n \rightarrow (14)$$

The continuity equation for two dimensional steady flow,

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

and writing the corresponding central difference equations centered around point (i,j)

$$\frac{(\rho u)_{i+\frac{1}{2},j} - (\rho u)_{i-\frac{1}{2},j}}{\Delta x} + \frac{(\rho v)_{i,j+\frac{1}{2}} - (\rho v)_{i,j-\frac{1}{2}}}{\Delta y} = 0 \rightarrow (15)$$

Refer eqns (13) & (14) to expand eqn (15) and drop superscripts.

$$\frac{(\rho u^*)_{i+\frac{1}{2},j} - \frac{\Delta t}{\Delta x}\left(P'_{i+1,j} - P'_{i,j}\right) - (\rho u^*)_{i-\frac{1}{2},j} + \frac{\Delta t}{\Delta x}\left(P'_{i,j} - P'_{i-1,j}\right)}{\Delta x} + \frac{(\rho v^*)_{i,j+\frac{1}{2}} - \frac{\Delta t}{\Delta y}\left(P'_{i,j+1} - P'_{i,j}\right) - (\rho v^*)_{i,j-\frac{1}{2}} + \frac{\Delta t}{\Delta y}\left(P'_{i,j} - P'_{i,j-1}\right)}{\Delta y} = 0$$

Rearranging above eqn,

$$\frac{1}{\Delta x}\left[(\rho u^*)_{i+\frac{1}{2},j} - (\rho u^*)_{i-\frac{1}{2},j}\right] + \frac{1}{\Delta y}\left[(\rho v^*)_{i,j+\frac{1}{2}} - (\rho v^*)_{i,j-\frac{1}{2}}\right]$$

$$+ P'_{i,j}\left[\frac{2\Delta t}{\Delta x^2} + \frac{2\Delta t}{\Delta y^2}\right] - \frac{\Delta t}{\Delta x^2}\left[P'_{i+1,j} + P'_{i-1,j}\right] - \frac{\Delta t}{\Delta y^2}\left[P'_{i,j+1} + P'_{i,j-1}\right] = 0$$

In above eqn.

$$a = \left[\frac{2\Delta t}{\Delta x^2} + \frac{2\Delta t}{\Delta y^2}\right]; \quad b = \frac{-\Delta t}{\Delta x^2}; \quad c = \frac{-\Delta t}{\Delta y^2};$$

$$d = \frac{1}{\Delta x}\left[(\rho u^*)_{i+\frac{1}{2},j} - (\rho u^*)_{i-\frac{1}{2},j}\right] + \frac{1}{\Delta y}\left[(\rho v^*)_{i,j+\frac{1}{2}} - (\rho v^*)_{i,j-\frac{1}{2}}\right]$$

$$\therefore \boxed{a P'_{i,j} + b\left(P'_{i+1,j} + P'_{i-1,j}\right) + c\left(P'_{i,j+1} + P'_{i,j-1}\right) + d = 0} \quad \rightarrow (16)$$

Eqn (16) is the pressure correction formula.

The SIMPLE Algorithm

The acronym SIMPLE stems from semi-implicit method for pressure-linked equations. The semi-implicit terminology refers to the arbitrary setting of A', B', $(\rho u')^n$ and $(\rho v')^n$ equal to zero, thus allowing the pressure correction formula to have P' appearing at only four grid points.

If this artifice had not been used, the resulting pressure correction formula would have included velocities at neighboring grid points. Those velocities are in turn influenced by pressure corrections in their neighborhood, and the resulting pressure correction formula would have reached much further

into the flow field, essentially coupling the entire pressure correction field in one equation. This would have represented a "fully implicit" equation.

Instead, because of the above artifice, the pressure correction formula contains p' at only four grid points, and hence it is termed as only semi-implicit.

The step-by-step procedure for the SIMPLE algorithm is,

1. Keeping in mind the staggered grid, guess values of $(p^*)^n$ at all the "pressure" grid points. Also arbitrarily set values of $(\rho u^*)^n$ and $(\rho v^*)^n$ at the proper "velocity" grid points. Here, we are considering the grid points internal to the flow field; the treatment of points on the boundaries will be discussed later.
2. Solve for $(\rho u^*)^{n+1}$ and $(\rho v^*)^{n+1}$ in the following equations at all appropriate internal grid points.

$$(\rho u^*)^{n+1}_{i+1/2,j} = (\rho u^*)^{n}_{i+1/2,j} + A^* \Delta t - \frac{\Delta t}{\Delta x}\left(p^*_{i+1,j} - p^*_{i,j}\right)^n$$

$$(\rho v^*)^{n+1}_{i,j+1/2} = (\rho v^*)^{n}_{i,j+1/2} + B^* \Delta t - \frac{\Delta t}{\Delta y}\left(p^*_{i,j+1} - p^*_{i,j}\right)^n$$

3. Substitute these values of $(\rho u^*)^{n+1}$ and $(\rho v^*)^{n+1}$ in pressure correction formula and solve for p' at all interior grid points.

4. Calculate p^{n+1} at all internal grid points from this equation

$$p^{n+1} = (p^*)^n + p'$$

5. The values of p^{n+1} obtained in step 4 are used to solve the momentum equations again. For this, we designate p^{n+1} obtained above as the new values of $(p^*)^n$ to be inserted into the equations presented in step 2. with this interpretation, repeat steps 2 to 5 until convergence is achieved. A reasonable criterion to use for a measure of convergence is when the mass source term 'd' approaches zero.

When convergence is achieved, the velocity distribution has been obtained which satisfies the continuity equation. The whole function of the pressure correction formula is to aim the iteration process in such a direction, that when the velocity distribution is calculated from the momentum equations, it will eventually converge to the correct distribution which satisfies the continuity equation.

On a related matter, the pressure correction formula may exhibit a divergent behavior, for such case in Step 4 following equation is used.

$$p^{n+1} = (p^*)^n + \alpha_p p'$$

where,

$\alpha_p \rightarrow$ underrelaxation factor about 0.8.

It may also be helpful in some cases to underrelax the value of u^* and v^* obtained from the equations in step 2.